Chapter 2

Interest rates

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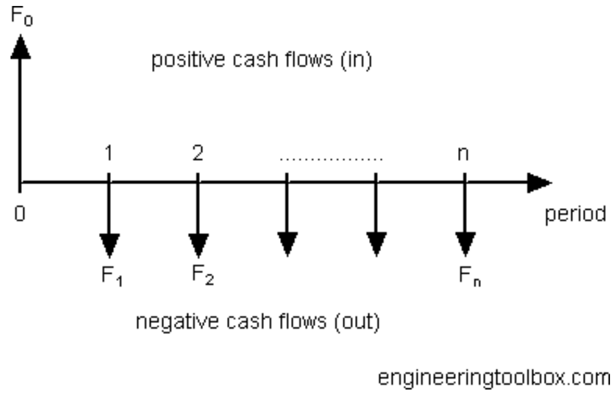
An interest rate measures the price to pay for borrowing money.

I) Simple interest rate

We consider a unit of time, usually a year Y.
An amount of money is borrowed at the simple interest rate r (per year) if the borrower must pay the lender, for each year of loan, the amount $I = S * r$ where S is the amount originally borrowed.
S is often called the "nominal" or "principal".
As a consequence, after N years, the borrower has paid $N * S * r = \text{Total Interest}$
and must repay the principal S.
This is summarized in a diagram of cash flow.



II) Compound interest

The idea here is that the interests are not paid immediately by the borrower, but become part of the amount borrowed, so that they contribute to the calculation on the next period.
Consider a loan of the amount S at time 0, with interest rate r (per year), coupounded annually.

- At time 0, you borrow S.
- At time 1, the interest on S is $I = r * S$
- At time 2, the interest over the period 1→2 is $r(S + rS) = r(1 + r)S$
- At time 3, the interest over the period 2→3 is $r((1 + r)S + r(1 + r)S) = r(1 + r)^6 S$

So the total amount borrowed at time 3 is $(1 + r)^6 S + r(1 + r)^6 S = (1 + r)^7 S$
At time N, the debt of the borrower is now $(1 + r)^8 S = F_8$.

Vocabulary

$$S = \text{principal or initial/present value of the loan}$$
$$F_8 = \text{future value of the loan at time } N$$

III) Interest compounded over short periods/continuous compounding

Notations:

- P will be the principal
- r the annual interest rate (sometimes called nominal interest rate)
- m the number of compounding periods per year ($m = 12$ for monthly compounding, $m = 360$ for daily compounding)

- t the number of years
- $i = \frac{J}{K}$ the interest rate per compounding period
- $n = mt$ the number of compounding periods over t years

- P_M the future value after n periods

We saw in the last part that : $P_M = (1+i)^M P$ and that the future value after t years : $P_{KN} = \left(1+\frac{J}{K}\right)^{KN} P$

In particular, for $t = 1$, $P_K = \left(1+\frac{J}{K}\right)^{K} P$.

What is the interest rate r_{OPP} per year, with period of compounding of 1 year, which gives the same future value ?

We have to solve $(1+r_{OPP})P = \left(1+\frac{J}{K}\right)^{K} P$:

$$1 + r_{OPP} = \left(1+\frac{r}{m}\right)^{K}$$

$$r_{OPP} = \left(1+\frac{r}{m}\right)^{K} - 1$$

which is called the effective interest rate per year.

Continuous compounding consists in letting $m \to +\infty$: extremely short compounding periods

$$P_{KN} = \left(1+\frac{r}{m}\right)^{KN} P \to_{K\to TU} e^{JN}P$$

Proof: take $\ln\left(\frac{W_{XY}}{W}\right) = mt * \ln\left(1+\frac{J}{K}\right)$

We know that $\lim_{Z\to[} \frac{\backslash](^{TZ}}{Z} = 1$

Take $x = \frac{1}{K}$ then $\ln\left(\frac{W_{XY}}{W}\right) = rt * \frac{\hat{}}{Z}\ln(1+x) \to_{Z\to[} rt$

And since exp is continuous, $\frac{W_{XY}}{W} \to_{Z\to[} e^{JN}$.

So if we have continuous compounding at the nominal annual rate r, the future value after t years is $e^{JN}P$ (P = principal).

What is the effective annual rate r_{OPP} in case of continuous compounding?

We must have $(1+r_{OPP})P = e^{J}P$:

$r_{OPP} = e^{J} - 1$

<u>Some useful formulas</u>

<u>Summations:</u>

Arithmetic series: $a_M = \alpha n + \beta$

$$S_M^d = \sum_{ef[}^{M} a_e = \alpha\left(\sum_{ef\hat{}}^{M} k\right) + (n+1)\beta = \frac{\alpha n(n+1)}{2} + (n+1)\beta$$

Geometric sum: $S_M^h = \sum_{ef[}^{M} g_e \qquad g_e = R^e g_[$

$$S_M^h = \frac{g_[(R^{MT\hat{}} - 1)}{R-1} \quad \text{if } R \neq 1$$

$$S_M^h = (n+1)g_[\quad \text{if } R = 1$$

What about other sums, like

$$\tilde{S} = \sum_{ef[}^{M} kR^e g_[$$

Trick of generating functions:

It is to consider the sum $\tilde{S}$ as a function of R. It is a polynomial. We can write

$$\tilde{S}(R) = R\left(\sum_{ef\hat{}}^{M} kR^{e\mathbf{1}\hat{}}\right) g_[$$

$$\tilde{S}(R) = Rg_[\left(\sum_{ef[}^{M} R^e\right)^{\mathrm{m}} = Rg\left(\frac{R^{MT\hat{}} - 1}{\ldots}\right)^{\mathrm{m}} \quad \text{for } R \neq 1$$

$[\quad R-1$

$$= \frac{Rg_{[} \quad n+1\, R^{\mathrm{M}}\, R-1-R^{\mathrm{MT}^{\wedge}}-1}{R-1^{\,6}}$$

$$\tilde{S}(R) = g_1 R \left(\frac{nR^{n+1} - (n+1)R^{n} + 1}{(R-1)^2} \right) \quad \text{for } R \neq 1$$

For $R = 1$, $\tilde{S}(1) = \sum_{k=1}^{n} k g_1 = \frac{n(n+1)}{2} g_1$