Chapter 6

Derivatives

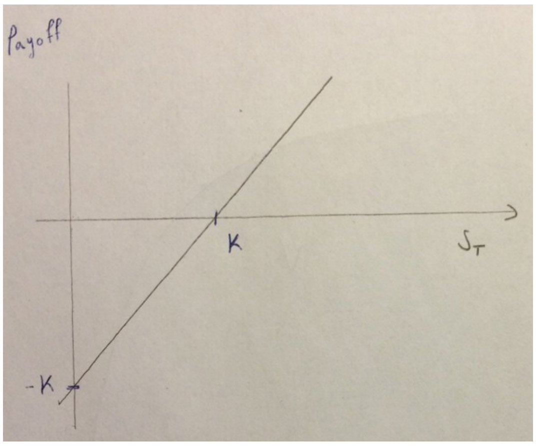
Definition: a derivative is a contract bond on an underlying object, which can be a currency exchange rate, an interest rate, a bond, a stock, or any kind of security.

I) Futures and forwards

Definition: in a future/forward contract, two parties A, B decide the following at time 0:
At time $T > 0$, A will buy from B one unit of a specified underlying asset X at a fixed price K.
Note that T, the asset X and the price K are known at time 0.
What is unknown is the price S_T of the asset X at time T on the market. By assumption, S_T is unpredictable.

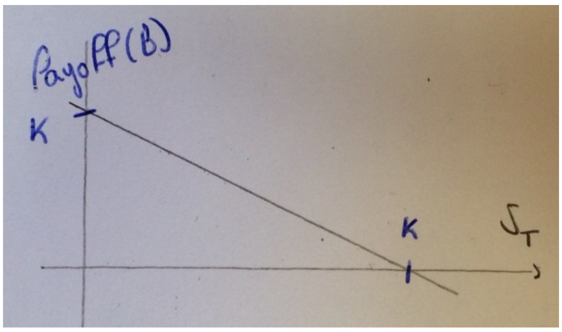
Note that this contract is symmetrical: A will have to buy X from B at price K at time T and B will have to sell X to A at price K at time T.

Notions of payoff: The payoff for A at time T will be $S_T - K$.



Since S_T is unpredictable, $Payoff(A)$ is unpredictable too.

$Payoff(A)$ is the net cash flow that A will get buying X from B at price K and selling it immegiately on the market at price S_T.
Similarly, the $Payoff(B)$ at time T is $K - S_T$.



This is the cash flow that B will get, buying X on the market at price S_T and selling it to A at price K.

Question: Can one compute the value of a forward contract for A or B? In other words, should A or B pay a compensation to B or A, at time 0?

At time 0, the future value S_T of the underlying is unknown, but the present market value S_0 of the underlying is known.

Proposition: In the absence of arbitrage opportunity, the present value at time 0 of the forward contract for A is $PV_0(A) = S_0 - (1 + r)^{6)} K$ where r is the interest rate per compounding period for an investment without risk on the market (which is assumed to be the same as the interest rate for loans).

In other words, if $S_0 - (1+r)^{6)} > 0$, then A should pay $P = S_0 - (1+r)^{6)}K$ to B.
If $S_0 - (1+r)^{6)}K < 0$ then B should pay $(1+r)^{6)}K - S_0$ to A (at time 0).

Proof: Let P be the price paid by A to B, in order to sign the contract at time 0.
We want to show that :

1. $P_0 = S_0 - (1+r)^{6)}K$.
   A can do the following:
   - At $t = 0$:
     - Sign the contract with B and pay P_0 to B.
     - Take a short position on the underlying: $Cashflow = S_0$.
     - Invest the amount $(1+r)^{6)}K$ (with no risk) at interest rate r.

     The net cash flow of A at time 0 is then: $-P_0 + S_0 - (1+r)^{6)}K > 0$.
   - At $t = T$:
     - Buys the underlying from B at price K: $Cashflow = -K + P_)$.
     - Choses the short position on the underlying ⇔ give back the underlying at price $P_)$.
     - Choses the investment at rate r: $Cashflow = K$

     The net cash flow of A at time T is then: $(-K + P_)) - (P_) + K) = 0$.

   This is an arbitrage in the generalized sense. So we cannot have : $P < S_0 - (1+r)^{6)}K$.
2. If $P_0 > S_0 - (1+r)^{)}K$. Then B can make exactly the opposite of what A does in case 1: → arbitrage for B.
   If $P_0 > S_0 - (1+r)^{6)}K$.
   B can do the following:
   - At $t = T$:
     - Sells the underlying to A at price K and buy it immediately: $Cashflow = K - P_)$
     - Choses the long position on the underlying ⇔ selling the underlying at price $P_)$.
     - Reimburses the loan at rate r: $Cashflow = -K$

Definition: The forward price of the underlying X at time 0 is the amount K such that

$$S_0 - (1+r)^{6)}K_{@ABCDBE} = 0 \qquad \Leftrightarrow \qquad K_{@ABCDBE} = (1+r)^{)} * S_0$$

If the forward contract is such that $K = K_{@ABCDBE}$, then $P_0 = 0$: there is no cash flow between A and B at time 0.

II) Options

An option is a contract between two parties A and B, which is not symmetrical. As in a forward, there is an underlying asset X.
We will only consider European options, for which the future date T of exercise of the option is fixed.
There are two kinds of European options:

- Call options: At time 0, A pays an amount to B, and gets the rights to buy the underlying X from B at time T, at strike price K (but maybe A will decide not to buy X). B receives P_0 at time 0, but at time T, if A decides to buy X, B will be forced to sell it at price K. In that case, we will say that the option is "exercised".
  We say that the buyer A of the call option is "long" on the call option, and that the seller B is "short" on the call option.
- Put options: At time 0, A pays an amount $P_0 > 0$ to B, and gets the right to sell X to B at time T and strike price K (but maybe A will decide not to sell X). B receives P_0 but will be forced to buy X from A at price K at time T. A is long, B is short on the put option.

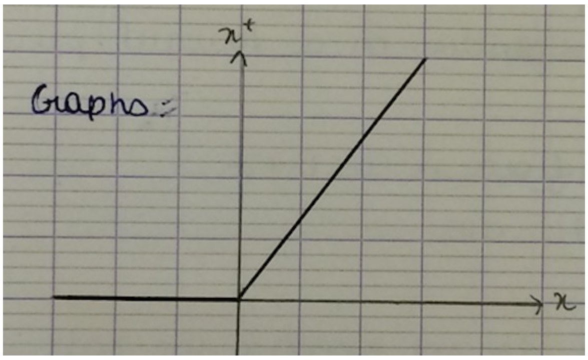
At time T, the value of the underlying asset is S_T. The “strike price” is K.

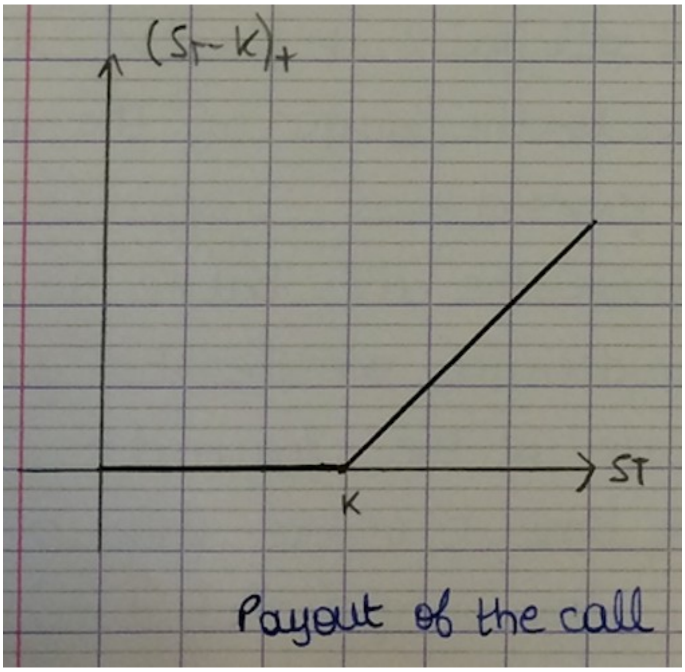
- “call/put options” : it is interesting to exercise a call/put in the case $S_T > K$; $S_T < K$. In this case, the call is “in the money”.
- If $S_T = K$, the call/put is “at the money”

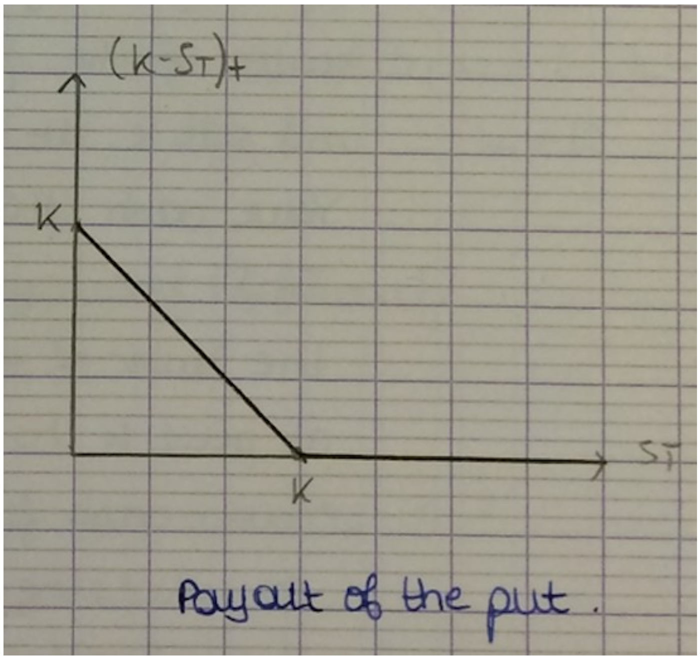
- If $S_T < K$, the call/put is "out of the money" : you should not exercise the option.

In terms of value, the value of the call/put at time T is zero if $S_T < K$; $S_T \geq K$ and equal to $S_T - K > 0$; $K - S_T > 0$ when $S_T > K$; $S_T < K$.
In all cases, the value or payout of the call/put at time T is $(S_T - K)_+$; $(K - S_T)_+$ where $x_+ = x$ if $x \geq 0$, $x_+ = 0$ if $x \leq 0$.

Graphs :







Question: How much should one pay for a call? For a put?

Theorem (put-call parity):
Let C_0, P_0, V_0 be the values at time 0 of a call, a put, and a forward on the same underlying asset X, with same $T > 0$, same $K > 0$.
In the absence of arbitrage opportunity:

$$C_0 - P_0 = V_0$$

Proof:
Assume for instance that $C_0 - P_0 < V_0$.
At time 0, you can:

- Take a long position on a call
- Take a short position on a put
- Take a short position on a forward

All of those actions, at the underlying asset X at strike price K.

The resulting cash flow is – $C_0 + P_0 + V_0$.

At time T,

- If $S_T \leq K$: you do not exercise the call. But the buyer of the put unit sells you X at price K, and you will sell X at price K to the buyer of the forward. Your cash flow is – $K + K = 0$.
- If $S_T > K$: you exercise the call, but the buyer of the put does not exercise it: you sell X at price K to the buyer of the forward.
  Your cash flow at time T is – $K + K = 0$.

So, whatever S_T is, your cash flow at time T is zero, this is an arbitrage.

Similarly, if $C_0 - P_0 > V_0$, at time 0, you can:

- Take a short position on a call
- Take a long position on a put
- Take a long position on a forward

All of those actions, on the underlying asset X at strike price K.
The resulting cash flow is $C_0 - P_0 - V_0 > 0$.

At time T,

- If $S_T \geq K$: you do not exercise the put. But the buyer of the call will decide to buy X from you at price K, and you will buy X at price K from the seller of the forward. Your cash flow is $K - K = 0$.
- If $S_T < K$: you exercise the put, but the buyer of the call does not exercise it: you buy X at price K to the seller of the forward.
  Your cash flow at time T is $K - K = 0$.

So, whatever S_T is, your cash flow at time T is zero, this is an arbitrage.

Explanation of the end of the proof:
Behind the proof is the equality : $x_+ - (-x)_+ = x \quad \forall x \in \mathbb{R}$

$$\Rightarrow (S_T - K)_+ - (K - S_T)_+ = S_T - K$$

In other words, at time T:

$$Payoff(call) - Payoff(put) = payoff(forward)$$

This holds for any value of S_T.

The remaining question is : how can we compute C_0?

<u>Naïve approach (which fails):</u>
Suppose we can only trade the option and the asset X at times 0 and T. One can also lend/borrow cash between $t = 0$ and $t = T$ at interest rate $r > 0$.

$$\begin{pmatrix} \text{Proof of } x_+ - (-x)_+ = x \\ \text{If } x \geq 0, x_+ = x \text{ and } -x \leq 0 \text{ hence } (-x)_+ \leq 0 \\ \text{Then } x_+ - (-x)_+ = x - 0 = x \\ \text{If } x \leq 0, x_+ = 0 \text{ and } -x \geq 0 \text{ hence } (-x)_+ = -x \\ \text{Then } x_+ - (-x)_+ = 0 - (-x) = -(-x) = x \end{pmatrix}$$

At time 0, the asset X has a value S_0.
At time T, it can have three possible different values, each with probability of $\frac{1}{3}$.

Example:

$$S_0 = 50 \begin{cases} p = \frac{1}{3}: & S_b = 110 * payoff : 110 - 60 = 50 \\ p = \frac{1}{3} & S_b = 70 * payoff : 70 - 60 = 10 \\ p = \frac{1}{3} & S_b = 40 * payoff : 0 \end{cases}$$

Let us take $K = 60$.
At time 0, you cannot predict S_b, so you cannot predict the payoff of the call. But you can compute an expected payoff.

$$E[payoff] = \frac{1}{2} * 50 + \frac{1}{3} * 10 + \frac{1}{3} * 0 = 20$$

The present value at time 0 of this payoff is :

$$\frac{E[payoff]}{1 + r}$$

If $\frac{b}{bIB} = 0{,}95$ we get $\cong 19$ as price of the call.
This price is wrong, we will see that it allows arbitrage.

Indeed, at time 0 you could:
- Sell one call option
- Borrow 31 at interest rate r
- Buy one underlying asset

At time 0:
→ Net cash flow :

$$C_0 + 31 - S_0 = 19 + 31 - 50 = 0$$

At time 1:
- If $S_b = 110$ or 70 the call option is exercised: you have to sell the asset at price $K = 60$, and you reimburse $31 * (1 + r)^{cb} \underset{0,no}{\cong} 32{,}63$

→ Net cash flow:

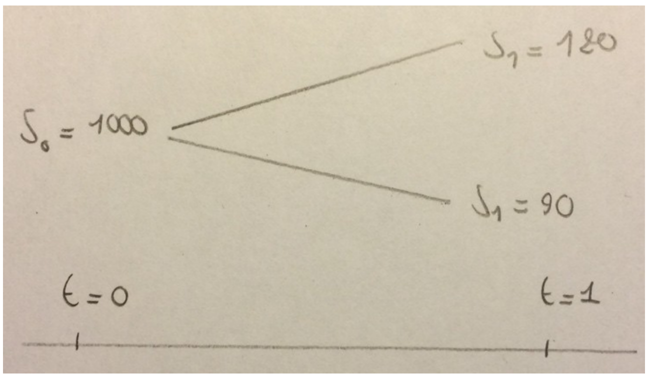
$$60 - 32{,}63 > 0$$

- If $S_b = 40$, the option is not exercised. You sell your asset on the market at price 40 and you reimburse 32,63

→ Net cash flow:

$$40 - 32{,}63 > 0$$

III) The binomial model with one time step



Suppose that cash can be borrowed (or lent) on the Market at interest rate r.
Les $K = 110$ be the strike price of a call option on the asset.
What is the no arbitrage price of this option at time 0?

In order to find this price, let us try to “replicate” the option.
Suppose that at time 0, you buy λ units of the underlying asset and you make a deposit of μ (euros) which will grow at rate r (hence you lend without risk).
Hence, $\lambda, \mu \in \mathbb{R}$:

At time 0, your net cash flow is:

$$-\lambda S_0 - \mu = -100\lambda - \mu$$

At time 1, you receive the reimbursement:

$$\mu(1+r) = 1{,}1\mu$$

- If $S_b = 120$, you sell your λ units of asset and receive:
$$S_b\lambda = 120\lambda$$
- If $S_b = 90$, you sell your λ units of asset and receive:
$$S_b\lambda = 90\lambda$$

So,

- If $S_b = 120$, your net cash flow is $120\lambda + 1{,}1\mu$
- If $S_b = 90$, your net cash flow is $90\lambda + 1{,}1\mu$

If instead you buy 1 unit of option at time 0, your cash flow at time 0 is – C_0.
At time 1:

- If $S_b = 120$ you buy the asset at price $K = 110$, you sell it in the market at $S_b = 120$ and your net cash flow is 10.
- If $S_b = 90$ you do nothing, your net cash flow is 90.

A replication must provide the same future cash flows as the call option, in every situation:

$$\begin{cases} 120\lambda + 1{,}1\mu = 10 & (1) \\ 90\lambda + 1{,}1\mu = 0 & (2) \end{cases}$$

Subtracting $(1) - (2)$, we get $30\lambda = 10 \Leftrightarrow \frac{b}{c}$

$(2) \Rightarrow \mu = -\frac{30}{1{,}1}\lambda = -\frac{30}{1{,}1}$

So the application corresponds to buying $\frac{b}{c}$ of asset and in borrowing $-\mu = \frac{c0}{b,b}$ € at interest rate r.

General principle: If two investment strategies deliver exactly the same ash-flows in the future, then their price at time 0 should be the same, otherwise an arbitrage would be possible.

Consequence: the no arbitrage price of the option is the price of the option at time 0:

$$C_{\mathrm{DBs}} = 100\lambda + \mu = \frac{100}{3} + \frac{30}{1{,}1}$$

One can give general formulas taking $r > 0$ arbitrarily and $S_b = uS_0$, or $S_b = dS_0$ with $0 < d < u$.
Assume that $dS_0 < K < uS_0$
If we buy λ units of asset and bon, u euros at time 0, then at time 1:

- If $S_b = uS_0$ you receive $S_0u\lambda + (1+r)\mu$
- If $S_b = dS_0$ you receive $S_0d\lambda + (1+r)\mu$

Hence

$$\begin{cases} S_0u\lambda + (1+r)\mu = uS_0 - K \\ S_0d\lambda + (1+r)\mu = 0 \end{cases}$$

and

$$\lambda = \frac{uS_0 - K}{S_0(u-1)}$$

$$u = -\frac{S_0\lambda(uS_0 - K)}{(1+r)S_0(u-d)} \qquad \mu = -\frac{d}{u-d} * \frac{(uS_0 - K)}{1+r}$$

<u>Risk neutral probability:</u>
Let us look for a probability law P_* on the issues uS_0 and dS_0 such that:

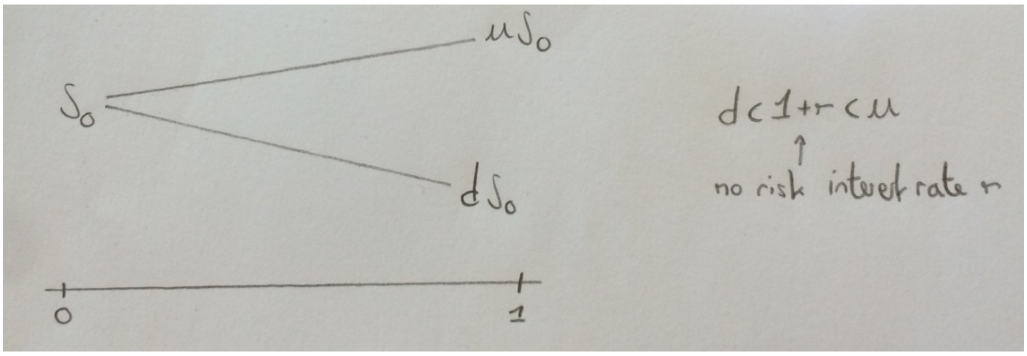
$$\frac{1}{1+r} E_{\mathrm{t}_*}[(S_{\mathrm{b}} - K)_{\mathbf{I}}\,] = C_{\mathrm{DBs}}$$

In other words, we look for P_*, probability law on the random variable S_{b} such that the present value of the expected payoff of the call is the same, as it’s a no arbitrage price.

We call P_* the risk-neutral probability law.
Let us compute $\mathbb{P}_*$. Let $P_* = \mathbb{P}_*(S_b = uS_0\ .)$
Then $\mathbb{P}_*(S_b = dS_0) = 1 - P_*$
Then $\frac{b}{bIB} E_{t_*}[(S_b - K)_I] = \frac{t^*}{bIB}(uS_0 - K)$
Hence the condition on P_* is : $\frac{t^*}{bIB}(uS_0 - K) = \frac{vw x6y}{v6E}\left(1 - \frac{E}{bIB}\right)$

$$P_* = \frac{1+r}{u-d}\left(1 - \frac{d}{1+}\right) = \frac{1+r-d}{u-d}$$

we must have $0 \leq P_* \leq 1$
Then, we must have $0 \leq 1+2-d \leq u-d \quad \rightarrow d < 1+r < u$
Note that these conditions must be satisfied, otherwise an arbitrage is possible.



The assumption $d \leq 1+r \leq u$ is necessary if we want to forbid arbitrage.
Indeed, if for instance, $1+r < d$. Then, at time 0, I can borrow the amount S_0 at rate r and buy one unit of asset at price S_0.
Net cash flow : $S_0 - S_0 = 0$

At time 1, I reimburse $(1+r)S_0$ and I sell the asset at price ≥ 0.
Net cash flow : $-(1+r)S_0 + dS_0 = (d - (1+r))S_0 \geq 0$

Moreover, the price of the asset is uS_0 with positive probability.
Net cash flow : $-(1+r)S_0 + uS_0 = (u - (1+r))S_0 > 0$

Similarly, if $u \leq 1+r$, I can do the opposite (lend S_0 and shot sell one unit of asset).

We saw that $\left.\begin{aligned} P^*(S_b = uS_0) &= P^* = \frac{(bIB)6E}{v6E} \\ P^*(S_b = dS_0) &= 1 - P^* = \frac{v6(bIB)}{v6E} \end{aligned}\right\}$ is the risk neutral probability for the call
Meaning that the price of the call on the asset at time 0,

$$C_0 = \underbrace{\frac{1}{1+r} E_{t_*}}_{\text{ÑÕÜáÜàâ äãåçÜ éè âêÜ ÜëÑÜíâÜì Ñãîéèè çàìÜö ÑÖéïãïåîâî ó}_*} ((S_b - K)_I) \qquad (S_b - K)_I \text{ is the payoff of the call at time 1}$$

Let us compute $\frac{b}{bIB} E_{t_*}(S_b) = \frac{b}{bIB}(P_* u + (1 - P_*)d)S_0 = \frac{((bIB)6E)vI(v6(bIB))E}{(bIB)(v6E)} S_0 = S_0$
So we have found $S_0 = \frac{b}{bIB} E_{t_*}(S_b)$
Finally, consider an amount A_0 invested at time 0 at rate r.
Then $\frac{b}{bIB} E[A_b] = \frac{b}{bIB}(1+r)A_0 = A_0$
Hence any contract based on the underlying asset, the option or the investment of cash will have the property:

$$v_0 = \frac{1}{1+r} E\, p_* \;\; v_{\mathrm{b}}$$

Note that this property implies that no arbitrage is possible.

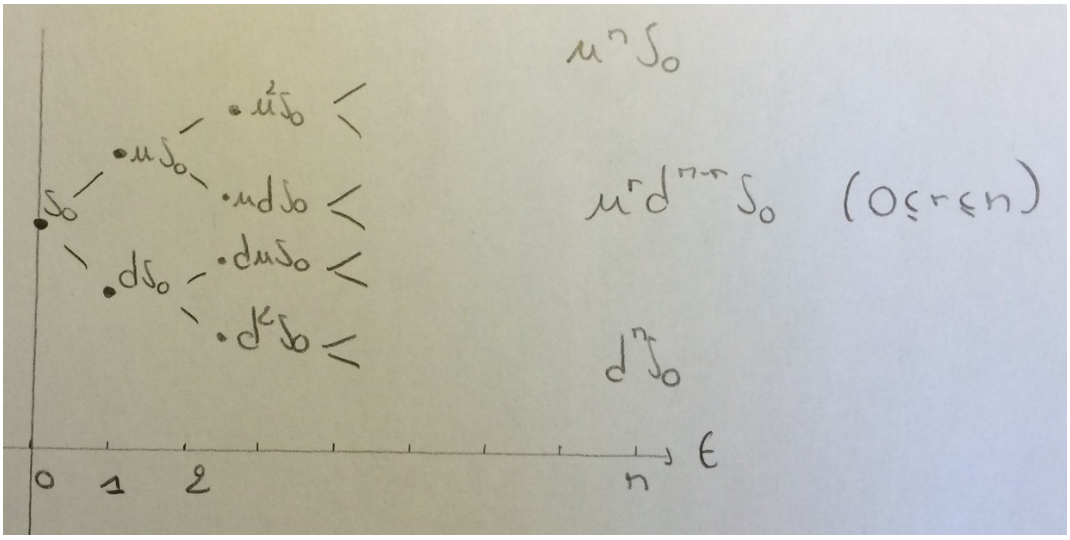
Indeed, an arbitrage would be a linear combination of contracts with the value $v_0 = 0$ at time 0. So $E_{P_*}[v_b] = 0$

Moreover, we cannot have $P_*(v_b < 0) = 0$ and $P_*(v_b > 0) > 0$

Hence we cannot have $\mathbb{P}(v_b < 0) = 0$ and $\mathbb{P}(v_b > 0) > 0$

No arbitrage is possible.

IV) Multi-time binomial model, underlying asset



We assume that, if S_i is known at time i, then S_{i+1}, can only take two values : uS_i or dS_i

$0 < d < (1 + r) < u$.

After n iterations, the possible values of S_n are $u^r d^{n-r} S_0$ where r is the number of lines the asset price varies from S_i to uS_i.

We assume that one can borrow a bond cash at rate r.

We consider a more general position. Obviously,

$C(n; n) = \begin{cases} S_n - K & \text{if } S_n \geq K \\ 0 \text{ otherwise} \end{cases} \quad : C(n, n) = (S_n - K)_+$

Question: What is the price $C(0; n)$ at time 0 of u European call on one unit of the asset, with exercise date r and strike price K?

We consider one more opened question:

Take $0 \leq i < n$

What is the price $C(i, n)$ at time i of a call on one unit of asset with exercise date n and strike price K?

One can determine $C(i, n)$ going backwards, from $i = n$ to $i = 0$.

Obviously, $C(n; n) = \begin{cases} S_n - K & \text{if } S_n \geq K \\ 0 & \text{otherwise} \end{cases} \quad : \quad C(n, n) = (S_n - K)_+$

What is $C(n - 1; n)$?

$$S_{n-1} = \begin{cases} S_n = uS_{n-1} \\ S_n = dS_{n-1} \end{cases} \Leftrightarrow 1 \text{ time step}$$

$$C(n - 1; n) = \frac{1}{1 + r} E_{P_*}[(S_n - K)_+] \underbrace{=} \frac{1}{1 + r} E_{P_*}[C(n; n)]$$

$$C(i;n) = \begin{cases} C_{\mathrm{V}}(1+i;n) \\ C_{\mathrm{E}}(1+i;n) \end{cases} \quad \ldots \quad \underbrace{C(n-1;n) = \begin{cases} C_{\mathrm{V}}(n;n) & \text{with probability } P_* \\ C_{\mathrm{E}}(n;n) & \text{with probability } 1-P_* \end{cases}}$$

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Hence we have:

$$C(i;n) = \frac{1}{1+r} E_{P_*}\left(C(i+1;n)\right)$$

$$C(n-1;n) = \frac{P_*}{1+r} C_u(n;n) + \frac{1-P_*}{1+r} C_d(n;n)$$

Hence at each step, knowing $C(i+1;n)$ in all cases, we can compute:

$$C(i;n) = \frac{1}{1+r} E_{P_*}\left(C(i+1;n)\right) = \frac{P_*}{1+r} C_u(i+1;n) + \frac{1-P_*}{1+r} C_d(1+i;n)$$

In fact, there is a global formula:

$$C(0;n) = \underbrace{\frac{1}{(1+r)^n}}_{\text{ìñáíéçàâ èãíâéö}} \underbrace{E_{P_*}[(S_n-K)_+]}_{\text{áëÑÜíâÜì Ñãîéèè ãâ âñ•Ü ù çàìÜÖ } P_*}$$

Note that under P_*, the event $\{S_n = u^k d^{n-k} S_0\}$ has probability:

$$\binom{n}{k} P_*^k (1-P_*)^{n-k}$$

$$\Rightarrow C(0;n) = \frac{1}{(1+r)^n} \sum_{k=0}^{n} \binom{n}{k} P_*^k (1-P_*)^{n-k} \left(u^k d^{n-k} S_0 - K\right)_+$$

Next time, we will replace times $1, 2, \dots, n$ by $\Delta T, 2\Delta T, n\Delta T = T$.

Price at time 0 of a European call option of exercise date n (time steps $0, 1, 2, \dots, n$) and strike price K, with underlying asset having value S_k at time k, with the behavior:

$$S_k = \begin{cases} uS_k \\ dS_k \end{cases} \qquad d < 1+r < u$$

$$C(0;n) = \frac{1}{(1+r)^n} E_{P_*}\left((S_n-K)_+\right) = \frac{1}{(1+r)^n} \sum_{k=0}^{n} \binom{n}{k} p_*^k (1-p_*)^{n-k} \left(S_0 u^k d^{n-k} - K\right)_+$$

$$p_* = \frac{1+r-d}{u-d}$$

Now, we take a time unit such that the time delay between step k and $(k+1)$ is ΔT, very small, with $T = n\Delta T$ fixed, as $n \to \infty$ (hence $\Delta T \to 0$).

Note that

$$\ln(S_{k+1}) = \begin{cases} \ln(S_k) + \ln(u) \\ \ln(S_k) + \ln(d) \end{cases}$$

Denote Y_k the random variable which equals $\ln(u)$ with probability p_* and $\ln(d)$ with probability $(1-p_*)$, with $Y_k = \ln\left(\frac{S_k}{S_{k-1}}\right)$ $(1 \le k \le n)$. We assume that $Y_1, \dots, Y_n$ are independent.

Then $\ln\left(\frac{S_n}{S_0}\right) = Y_1 + Y_2 + \dots + Y_n \sim \mathcal{B}(n, p_*)$ and $C(0;n) = \frac{1}{(1+r)^n} E_{P_*}\left((S_0 e^{Y_1+\dots+Y_n} - K)_+\right)$

Now, we assume that:

$$\begin{cases} \ln(u) = \mu\Delta T + \sigma\sqrt{\Delta T} \\ \ln(d) = \mu\Delta T - \sigma\sqrt{\Delta T} \end{cases} \Rightarrow \begin{cases} u = \exp(\mu\Delta T + \sigma\sqrt{\Delta T}) = 1 + \mu\Delta T + \sigma\sqrt{\Delta T} + \frac{\sigma^2\Delta T}{2} + o(\Delta T) \\ d = \exp(\mu\Delta T - \sigma\sqrt{\Delta T}) = 1 + \mu\Delta T - \sigma\sqrt{\Delta T} + \frac{\sigma^2\Delta T}{2} + o(\Delta T) \end{cases}$$

Moreover, the compounding period is now Δt, so the interest rate for this period is now $r\Delta T$ instead of r.

This gives a formula for $p_* = \frac{1+r\Delta T-d}{u-d} \cong \frac{1}{2} + \frac{1}{2} * \frac{r-\mu-\frac{\sigma^2}{2}}{\sigma}\sqrt{\Delta T} + O(\Delta T)$

By the central limit theorem, for n very large, $\frac{(Y_1+\dots+Y_n) - nE_{P_*}(Y_1)}{\sqrt{n}} \sim \mathcal{N}(0, \mathbb{V}(Y_1))$

We can compute $E_{P_*}(Y_1) = p_* \ln u + (1-p_*)\ln d$ and $\mathbb{V}(Y_1) = p_*(\ln u)^2 + (1-p_*)(\ln d)^2 - \left(E_{P_*}(Y_1)\right)^2$

As a result, one finds that for n very large, $\ln\left(\frac{S_T}{S_0}\right) \sim \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)T; \sigma^2 T\right)$

$$\frac{1}{\dots}$$

$$C(0;T) = \underbrace{e^{6B«}}_{b} E_{t_*}((S_) - K)_{\mathrm{I}}) = e^{6B)} * \qquad e^{¥¿^{º})\ À6\ B6_{¥¿}\ )} \qquad S_0 e^{À} - K\, dy$$

º åñ• »ø→x ø (b I Bæ))»ø

Black and Scholes formula for European option pricing.

Exercise : $\begin{cases} S_0 = 100 \\ u = 1{,}20 \\ d = 0{,}90 \\ r = 10\% \end{cases}$

$$S_0 = 100 \begin{cases} S_0 * 1{,}20 \quad et \quad C_v(1;2) \begin{cases} S_0 * 1{,}44 = 144 \Rightarrow C(2;2) = 144 - 106 = 38 \\ S_0 * 1{,}08 = 108 \Rightarrow C(2;2) = 2 \end{cases} \\ S_0 * 0{,}9 \quad et \quad C_E(1;2) \begin{cases} S_0 * 1{,}08 = 108 \Rightarrow C(2;2) = 2 \\ S_0 * 0{,}81 = 81 \Rightarrow C(2;2) = 0 \end{cases} \end{cases}$$

Call European option $K = 106$ à $t = 2$

$$p^* = \frac{1 + r - d}{u - d} = \frac{1{,}1 - 0{,}9}{1{,}2 - 0{,}9} = \frac{2}{3}$$

$$C_v(1;2) = \frac{1}{1{,}1}\left(\frac{2}{3} * 38 + \frac{1}{3} * 2\right) = \frac{1}{1{,}1} 26 = 23{,}63$$

$$C_E(1;2) = \frac{1}{1{,}1}\left(\frac{1}{3} * 2\right) = \frac{2}{3{,}3}$$

$$C(0;2) = \frac{1}{1{,}1}\left(\frac{2}{3} * 23{,}63 + \frac{1}{3} * 1{,}21\right)$$

Considérons maintenant une option américaine:

Strike price $K = 1{,}06$

Exercise date 1 or 2

Si l'option est exercée à la date 2, son payoff est le même que pour l'option européenne.

A la date 1, dans l'état "u" où $S_b = 120$, si j'exerce le call je gagne $(120 - 106) = 14 < C_v(1;2) = 23{,}63$: il vaut mieux attendre et $C^{\text{œ}}_v(1;2) = C^{\neg}_v(1;2)$

Dans l'état "d" où $S_b = 90$ je n'ai aucun intérêt à exercer l'option à la date 1, et $C^E_{\text{œ}}(1;2) = C^E_{\neg}(1;2)$

Ici le call américain équivaut au call européen.